# A well-balanced DG scheme for the resolution of the shallow water equations

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**Abstract** In this work, a novel framework for the construction of Discontinuous Galerkin (DG) schemes using augmented Riemann solvers for the resolution of the Shallow Water Equations (SWE) is proposed. The benefits of this family of solvers lie in the exact preservation of the Rankine Hugoniot (RH) condition at cell interfaces at the discrete level, ensuring the preservation of equilibrium solutions (i.e. the well-balanced property) without requiring extra corrections of the numerical fluxes.

**Keywords:** shallow water, augmented solver, high order, source terms, Discontinuous Galerkin

## 1. Introduction

The1D, $x$-split SWE can be expressed as

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \frac{\partial}{\partial t}\begin{pmatrix} hu \\ hu^2 + 0.5gh^2 \\ huv \end{pmatrix} = \begin{pmatrix} 0 \\ -ghz_x \\ 0 \end{pmatrix}$$

with $h$ the water depth, $hu$ and $hv$ the discharges, $g$ the acceleration of gravity and $z$ the bed elevation.

## 2. Numerical method

The DG method is based on the approximation of the solution by a linear combination of basis functions $\{\phi_l\}_{l=0,\dots,N_d}$ as follows

$$\boldsymbol{U}(x,t) \approx \boldsymbol{U}_i(x,t) = \sum_{l=0}^{N_d} \widehat{\boldsymbol{U}}_{i,l}(t)\phi_l(x)$$

This allows to express the updating scheme for the unknown degrees of freedom as $\widehat{\boldsymbol{U}}_{i,l}(t)$

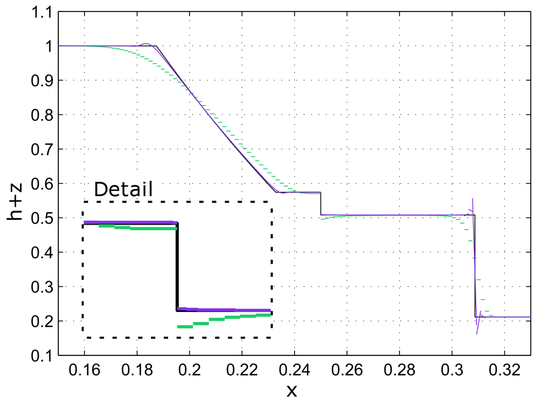
$$\frac{\partial \widehat{\boldsymbol{U}}_{i,k}}{\partial t} = -\frac{1}{a_k}\left(\int_{\Gamma_i} \phi_k \mathcal{F}^{*,-} d\Gamma - \int_{\Omega_i} \boldsymbol{F}\cdot\frac{\partial \phi_k}{\partial x} dx - \int_{\Omega_i} \phi_k \boldsymbol{S} dx\right)$$

where $\boldsymbol{F}$ is the matrix of physical fluxes, $\mathcal{F}^{*,-}$ stands for the numerical flux in the $x$-direction and $\boldsymbol{S}$ is source term.

## 3. Numerical results

Case 1 is an evaluation of the convergence rates of the well-balanced DG3-SSPRK3 scheme with smooth topography. Table 1 shows its error norms and convergence rates.

Cases 2 and 3 are Riemann Problems, designed as a dam break the first one and as an equilibrium representing an ideal shear layer (contact wave) the second one.




The solution for the water surface elevation of Case 2 is plotted in Figure 1. The intermediate states are accurately predicted, as well as the location of the shock wave and rarefaction wave.

**Figure 1**: Case 2.

Both cases are computed by the well-balanced first order Godunov's scheme (green line) and DG2-SSPRK3 (purple line), with exact solution (black solid line), including in Case 3 the analytical solution (dashed line).

| Label | $h_L$ | $h_R$ | $hu_L$ | $hu_R$ | $hv_L$ | $hv_R$ | $z_L$ | $z_R$ | T | N |
|---|---|---|---|---|---|---|---|---|---|---|
| RP1 | 1.0 | 0.1614 | 0.0 | 0.0 | 1.0 | 0.1614 | 0.0 | 0.05 | 0.02 | 320 |
| RP2 | 1.0 | 1.0 | 0.0 | 0.0 | 1.0 | 2.0 | 0.0 | 0.0 | 0.032 | 160 |

**Table 2**: Initial condition for Cases 2 and 3. Units in $m$, $m^2/s$ and $s$.

In Case 3, Figure 2 shows that the 1-st order scheme adds a large amount of numerical diffusion, which would involve a strong mixing between the left and right regions in the context of the resolution of a shear layer.

The DG2-SSPRK3 scheme reproduces a sharper transition across the contact discontinuity. Its performance is much better than the first order version of the augmented Riemann solver and even than other higher order schemes.

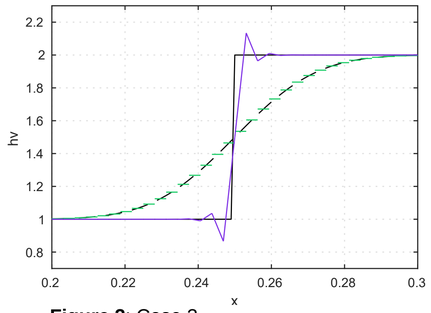



**Figure 2**: Case 3.

| | $h$ (water depth) | | | | $hu$ (discharge) | | | |
|---|---|---|---|---|---|---|---|---|
| Meshes ($N_j/N_{j+1}$) | $L_1(h)$ | $O_{L_1}$ | $L_\infty(h)$ | $O_{L_\infty}$ | $L_1(h)$ | $O_{L_1}$ | $L_\infty(h)$ | $O_{L_\infty}$ |
| 160/320 | $4.17\cdot10^{-09}$ | 3.48 | $2.97\cdot10^{-08}$ | 3.54 | $1.86\cdot10^{-08}$ | 3.48 | $1.40\cdot10^{-07}$ | 3.57 |
| 320/640 | $4.72\cdot10^{-10}$ | 3.14 | $3.30\cdot10^{-09}$ | 3.17 | $2.11\cdot10^{-09}$ | 3.14 | $1.55\cdot10^{-08}$ | 3.17 |
| 640/1280 | $5.74\cdot10^{-11}$ | 3.03 | $4.00\cdot10^{-10}$ | 3.04 | $2.57\cdot10^{-10}$ | 3.03 | $1.88\cdot10^{-09}$ | 3.04 |
| 1280/2560 | $7.12\cdot10^{-12}$ | 3.01 | $4.93\cdot10^{-11}$ | 3.01 | $3.19\cdot10^{-11}$ | 3.01 | $2.33\cdot10^{-10}$ | 3.01 |

**Table 1**: Convergence rates for Case 1.

## 4. Conclusions

DG schemes are automatically well-balanced when using augmented solvers, provided:

- an exact quadrature rule for the surface and volume integrals,
- and the discrete equilibrium satisfied by the reconstructed data.

The proposed scheme achieves the prescribed convergence rates and preserves the equilibrium states of relevance with machine precision.

Immediate goals:

- limitation of shock oscillations,
- correct resolution of dry/wet fronts
- parallelization of the 2D scheme using CUDA/OpenACC to solve it in the GPU.

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